Real Analysis and Topology
Assignment 7 — Topology and Continuity

Due Tuesday, December 7

Please do all the problems below. If you can't solve a problem, please show as much as you were able to do, so that I can indicate how you might go further. Feel free to discuss problems with others, but turn in work that you can call your own.

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1. Is it necessarily true that $C = f(f^{-1}(C))$ for any set C and any function f? Is it necessarily true that $D = f^{-1}(f(D))$? Explain.

2. Read Theorem 8.15 (on compositions of functions) and the half-proof given there. Then complete the other half of the proof.

3. Consider the function $f(x) = x^4$.

   (a) For any open interval (a,b), give a formula for the set $f((a,b))$. (Be careful to consider the different possible situations!)

   (b) Repeat (a) for the closed interval $[a,b]$.

   (c) For any open interval (c,d), give a formula for the set $f^{-1}((a,b))$.

   (d) Repeat (c) for the closed interval $[c,d]$.

   (e) Does this function satisfy the topological definition of continuity?

   (*NOTE:* If you try to prove that this function satisfies the ε-δ definition of continuity, you may gain an extra appreciation of the topological definition…)

   (f) If f is any continuous function and C is an arbitrary closed set, show that $f^{-1}(C)$ is also closed. Is this consistent with your answer to (d) above?

4. Suppose you have some function $f : A \to B$.

   (a) Is it possible for a set S (living inside of A) to have a smaller cardinality than its mapping $f(S)$? If so, provide an example; if not, prove it.

   (b) Is it possible for a set S (living inside of B) to have a smaller cardinality than its *inverse* mapping $f^{-1}(S)$? If so, provide an example; if not, prove it.

5. In class, we proved that a function satisfying the ε-δ definition of continuity satisfies the topological definition (in terms of mappings of open sets). Complete the proof that the two definitions are equivalent by proving that a function satisfying the topological definition must also satisfy the ε-δ definition.