Real Analysis and Topology
Assignment 6 — Cluster Points; Open and Closed Sets

Due Tuesday, November 30

Please do all the problems below. If you can't solve a problem, please show as much as you were able to do, so that I can indicate how you might go further. Feel free to discuss problems with others, but turn in work that you can call your own.

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1. What are the cluster points of the following sets?
   (a) $(0,1)\cup(3,4]$
   (b) $(0,1)\cup(1,2)\cup(2,3)\cup(3,4)\cup\cdots$
   (c) $\{1, 5, 1.9, 5.9, 1.99, 5.99, 1.999, 5.999, \cdots\}$
   (d) Can you construct a sequence $\{x_n\}$ with three cluster points? four? five? infinitely many?

2. Consider definitions (3) and (4) of "weakened neighborhoods" on p. 100 of the text.
   (a) Clearly (2) implies (3), but is it necessarily true that (3) implies (2)? If so, prove it; if not, find a counterexample (a set *S* that satisfies (3), but not (2)).
   (b) Does (3) imply (4)? Does (4) imply (3)?
   (c) Find a way to explain in ordinary English (in a way that a non-mathematician might understand) what kinds of sets satisfy conditions (3) and (4). Your explanation, if correct, will help you understand the book's cryptic remark that these definitions are not interesting. (HINT: both definitions say "**there exists**" an ε, a single quantity. It does not say that there must be arbitrarily small values of ε.)

3. Where in the proof of the Bolzano-Weierstrass Theorem did we use the Archimedean property?

4. (a) Which of the sets in Question 1(a)-(c) are open? Which are closed? Which are neither?
   (b) The **interior** of a set *A*, called A^0, is defined to be the union of all open intervals that are subsets of *A*. What are the interiors of the sets in Question 1(a)-(c)?
   (c) Explain why the interior of any set must be open.

5. Recall the Cantor set *C* from last week's midterm synthesis assignment, and recall especially the ***ternary*** expansion of members of *C*.
   (a) Prove that *C* contains no intervals; in other words, any interval contains members of the complement of *C*.
   (b) Recall from class that *C* is closed; hence it contains all of its cluster points. Show that *every* member of *C* is a cluster point of *C*. (This is an example of a ***perfect*** set, a set for which $S' = S$.)
   (c) Complete the proof, on the next page, that a denumerable set cannot be perfect. (It took me a while to see how to prove this. There may be a simpler proof… Take that as a challenge.) Notice that this implies that the Cantor set is uncountable!

Theorem: A denumerable set cannot be perfect.

Proof: Consider a denumerable set, *A*, whose members are all cluster points of *A*.

We shall find uncountably many cluster points of *A*, as follows: take any real number *x* from 0 to 1, and write out its binary "decimal" expansion (for instance, *x*=0.1100010101110…)

Take any two elements of *A*; by definition, they must be cluster points of *A*.

$\Downarrow$
∂
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Hence, we have a bounded interval [*a*, *b*] containing infinitely many elements of *A*.

$\Downarrow$
∂
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Now we have two disjoint closed sub-intervals of [*a*, *b*], both containing infinitely many elements of *A*.

If the first digit after the binary expansion of *x* is a 0, choose the leftmost of the two intervals; if it's a 1, choose the rightmost.

Now repeat the process above, finding two disjoint closed sub-intervals of the current interval, and choosing the leftmost or rightmost interval depending on the next binary digit of *x*.

$\Downarrow$
∂
$\Downarrow$

This establishes a series of nested, closed and bounded intervals, determined by *x*. Since … … …, each nested interval is at most half the size of the one before it in the chain.

So, by the Least Upper Bound Property, … … … … …

$\Downarrow$
∂
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We have now established a 1-to-1 correspondence between the interval [0,1] and a collection of cluster points of *A*. Hence *A* has uncountably many cluster points; but *A* is only countable. So, there must be cluster points of *A* that are not members of *A*.

Therefore, *A* is not perfect.

Q. E. D. !