Real Analysis and Topology
Assignment 4 — Neighborhoods, Bounds, Completeness

Due Thursday, November 4

Please do all the problems below. If you can't solve a problem, please show as much as you were able to do, so that I can indicate how you might go further. Feel free to discuss problems with others, but turn in work that you can call your own.

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1. (a) Show that both the union and intersection of two neighborhoods of x are also neighborhoods of x.
   (b) Is the above statement true if you replace "two" with "finitely many"?
   (c) Is the statement true if you replace "two" with "infinitely many"?

2. (a) Show that a finite set contains its own supremum as a member.
   (b) What are the supremum and infimum of the empty set? (answer carefully…)
   (c) Show that if S is non-empty, then $\sup(S) \geq \inf(S)$.

3. Consider what would happen if you augmented $\mathbb{Q}$ with the single number $\sqrt{2}$.

   (a) Explain why, if a field contains both $\mathbb{Q}$ and $\sqrt{2}$, it must contain any number of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$.

   (b) Prove that the above set is an ordered field.
   (c) Is it countable or uncountable? Explain.
   (d) Does it satisfy the Least Upper Bound property? The Archimedean property? The Nested Intervals property?

4. The **dyadic rational numbers** are the set of all rational numbers which, when reduced to lowest terms, have a denominator with a non-negative power of 2.
   (a) Is the set of dyadic rational numbers a field?
   (b) Is it an *ordered* field?
   (c) Which of the three properties in 3(d) does it satisfy?
   (d) Show that the dyadic rational numbers are dense in $\mathbb{R}$ (that is, there is at least one of them in any interval of real numbers).

5. Do Problem 6.2.3.

6. Prove Theorem 6.3.