Real Analysis and Topology
Assignment 3 — Fields, Orderings, and Induction

Due Tuesday, October 19

Please do all the problems below. If you can't solve a problem, please show as much as you were able to do, so that I can indicate how you might go further. Feel free to discuss problems with others, but turn in work that you can call your own.

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1. In any field, show that the only solutions of the equation $x^2 = x$ are 0 and 1. (Be careful with all these questions to make sure your steps are justified by the axioms!)

2. Consider any field with three elements: 0, 1, and something else, say, *x*. Construct addition and multiplication tables for this field as much as possible, following the laws of fields (give very short explanations for each entry in the tables). If you are able to fill in the tables completely, explain what this implies about your field (*Hint:* compare it to $\mathbb{Z}_3$).

3. (a) Are there any fields for which the additive identity and the multiplicative identity are the same element (i.e., 0 = 1)?
   (b) Assuming that the field has at least two elements, repeat (a).
   (c) Is it possible for a field to have addition and multiplication as identical operations (i.e., $\forall a, b,\ \ a + b = a \cdot b$)? What about a field with at least two elements?

4. Prove that the sum of the odd numbers up to $2n - 1$ is equal to n^2 (use induction).

5. Read Section 4.5 on strong induction.
   (a) Explain why "strong induction" is a good name for this procedure.
   (b) Explain why "strong induction" is a bad name for this procedure.
   (c) Let $x_1 = 1$, and let $x_n = \sum_{j=1}^{n-1} x_j$ for all $n > 1$. Show that all the members of this sequence, other than the first two, are even.
   (d) Show that all the members of the above sequence are powers of 2.

6. Consider the following argument that all the coins in any wallet are the same colour: If there is one coin in the wallet, the statement is clearly true. Suppose the statement is true for *n* coins, and consider the set of coins $\{x_1, x_2, x_3, \ldots, x_{n+1}\}$. By the induction hypothesis, the set of coins $\{x_1, x_2, x_3, \ldots, x_n\}$ are all the same colour; also by the same induction hypothesis, so are the coins $\{x_2, x_3, x_4, \ldots, x_{n+1}\}$ (there are, after all, *n* coins in this set). Since the sets overlap, all the coins must be the same colour.
   (a) Explain carefully why this argument is fallacious. For which value of *n* does the argument first fail?
   (b) Prove that all math students are brilliant.

7. (**optional**) Prove Theorem 4.12(b).