Real Analysis and Topology
Assignment 2 — More Continuity; Sets; Cardinality
Due Friday, October 8

Please do all the problems below. If you can't solve a problem, please show as much as you were able to do, so that I can indicate how you might go further. Feel free to discuss problems with others, but turn in work that you can call your own.

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1. (a) $f(x) = \dfrac{x+3}{x^2-25}$ (b) $f(x) = \dfrac{x^2-25}{x-5}$ (c) $f(x) = |x| - 1$
   One of the above functions is continuous for all *x*: which one? One can be made to be continuous by redefining it at one or two values of *x*; explain. The remaining function cannot be so redefined to make it continuous; explain in terms of the definition of continuity.

2. The **Dirichlet function**, often used as a counterexample in analysis, is defined as follows:
   $$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$
   (a) What would its graph look like, if plotted for $x = 0 \ldots 1$?
   (b) Is it continuous for any value of *x*? Explain.
   (c) Suppose that we alter *f* by replacing the 1 with x. Show that the revised function is continuous for one, and only one, value of *x*: namely, $x = 0$.

3. Prove the Triangle Inequality that we saw in the differentiability ⇒ continuity argument; namely, that $|x| - |y| \le |x - y|$ for all real *x*, *y*.

4. One of the following two statements is true; the other is false. Prove the statement that is true, and construct example sets that illustrate the falseness of the other.
   $((x \in A \cup B) \wedge (x \in A)) \Rightarrow (x \in B)$ $\qquad$ $A \subseteq (B \cup C) \Leftrightarrow (A \setminus C) \subseteq B$

5. Do Exercise 1.15.10.

6. Sometimes a set is defined to be infinite if it can be mapped, one-to-one, into a *proper* subset of itself.
   (a) Why might this definition be considered more satisfying than the definition we saw in class?
   (b) Use this definition to prove that $\aleph$ is infinite.
   (c) Use this definition to prove that $S = \{1,2\}$ is *not* infinite.

7. (**optional**) Is it true that every infinite set has a denumerable subset? Explain. (You may think you can prove it, but your proof will inevitably contain an apparently innocuous assumption known as the **axiom of choice**. It is now known that this axiom is undecidable.)